Equal Temperament

Some background information

Notes with two names

On a piano keyboard, the black note between the white notes G and A has two names: G sharp (G♯) and A flat (A ♭). This can be irritating. Then in written music that same note can appear in two different ways: as G♯ or A ♭ . This can be irritating. In the context of music in the key of E major the note will be called G♯, but in music in F minor the note will be called A ♭ . Similar comments apply to all the black notes on a piano.
[The proliferation of note-names does not stop there. C is also called B sharp. B is also called C flat. The note called C sharp or D flat is also known as B double-sharp. The note called B flat or A sharp is also known as C double-flat.]

One reason for all this is that in older systems of tuning the notes G♯ and A ♭ were very slightly different. When the modern system of tuning ("equal temperament"), such as is generally found on a piano for instance, became the main system of tuning for western music, and G sharp became the same note as A flat, then it was easiest not to change the way that notes are named or the way that music is written (using key signatures). The benefit of introducing a system where the black note between G and A had only one name would be small in comparison to the disadvantage of no longer using key signatures.

Older systems of tuning

The pitch of a note (how high or low it is) depends on how many vibrations per second are carried by the air to our eardrums. If you double the frequency (the number of vibrations per second), then the note becomes an octave higher. Whatever note you start with, **doubling the frequency makes the note an octave higher**. The frequency of "high doh" is double that of "low doh".

Now to be able to make tunes, we need a scale with enough notes between high doh and low doh; that is, we need notes with frequencies between one and two times the frequency of low doh. There are infinitely many numbers between 1 and 2; so there are infinitely many ways of choosing notes for our scale. The scale found on the highland bagpipes, for instance, differs noticeably from the scale used by orchestral instruments.

Some of the notes we need arise naturally. Now when we hear a single note from an acoustic musical instrument, although we perceive it as a single note there are always other higher and quieter notes present, which are called *overtones*. These overtones are the same notes for all musical instruments (though some overtones may be absent or too quiet to matter), but it is the pattern of loudness of the various overtones that is mainly what makes a note on a violin, say, sound different from the same note on a clarinet, say. We can make out the overtones more easily when the note is from a bright-sounding instrument.

The overtones have frequencies that are 2, 3, 4, 5, 6, … times the frequency of the note itself. Take as an example the note C, and suppose that we are trying to build up a scale based on C. The first overtone of C, having twice the frequency, must also be C, but an octave higher. The second overtone of C, having three times the frequency of C, must be higher still. Now since doubling the frequency goes up an octave, it follows that halving the frequency goes down an octave. Bringing the second overtone down an octave would give a note with frequency one-and-a-half times the frequency of our note C – and that means it is in the range where we are trying to make a scale. (In discussing these matters, we usually say 3/2 instead of one and a half.) This note with frequency 3/2 times the frequency of C is known as G, and the interval from C up to G is called a fifth. (The reasons for naming the notes and the intervals as we do would become apparent only after the scale has been constructed.)

Whatever note you start with, **multiplying its frequency by 3/2 gives you the note a fifth higher**. The general pattern is that whatever note you start with, multiplying its frequency by a fixed number changes the pitch of the note by a fixed interval.

So now we have low C (our original note), G and another C an octave higher; not much of a scale yet. The interval from G **up** to C is called a fourth, and we can now deduce what the frequency-multiplier is for an interval of a fourth. Here is the deduction.

For low C to high C, frequency is "times 2".
For low C to G, frequency is "times 3/2".
So for G to high C, frequency is "times what"?
That boils down to this. What do you have to multiply 3/2 by to get 2?
Well, 3/2 multiplied by 4/3 is 2.
So it's 4/3.

There are more deductions of this sort below, but with briefer explanations.
Whatever note you start with, **multiplying its frequency by 4/3 gives you the note a fourth higher**.

We can use this right away to find another note for the scale, the note a fourth higher than C. This note we call F and its frequency is 4/3 times the frequency of C.

Most of the old systems of tuning had the notes C, F and G as just described, though perhaps named differently. There were lots of ways of selecting further notes to complete a scale. Even for the major scale (C,D,E,F,G,A,B,C), there were slightly different versions of the notes D, E, A and B.

Pythagoras, famed for Pythagoras' Theorem, was more justly famous as the first person known to have analysed musical notes. Using a stringed instrument, he made overtones readily audible in the same way that a guitarist plays harmonics, and made deductions from his discoveries. He constructed the rest of the major scale like this. There was a need for smaller intervals to fill out the scale with more notes. He took the interval from F to G, which interval is called a tone, and applied it as often as necessary to put in notes between low C and F, and also between G and high C.
So Pythagoras' scale had the note D (as we call it) one tone above C, and then the note E (as we call it) one tone above D. In similar style, his scale had the note A (as we call it) one tone above G, and then the note B (as we call it) one tone above A. That completed his scale, the first known version of the major scale.
Since 4/3 times 9/8 is 3/2 (to go from F up to G), the frequency multiplier for Pythagoras' tone is 9/8. So in this scale, going from C to D multiplies the frequency by 9/8, and going from D to E again multiplies the frequency by 9/8. That means going from C to E multiplies the frequency by

9/8 times 9/8, which is 81/64.
His scale is impressively systematic. Now the interval from C to E is called a major third. So in Pythagoras' scale a major third has a frequency multiplier of 81/64. He has three major thirds (C to E, F to A and G to B) all with exactly the same interval. Except in two places, the interval between successive notes is always a tone: between E and F and also between B and C there is a smaller interval known as a hemitone (rather than a semitone). Perhaps you would like to calculate the frequency-multiplier for Pythagoras' hemitone; the answer is at the end of this article.

As mentioned, though, there were other versions of the major scale. As an example, here is a common way to choose the pitch of the note E. One of the overtones of the note C has a frequency five times that of C. This overtone we can call E. It is much too high to be in the octave where we are trying to build up a scale. So let's bring it down an octave by halving its frequency. That would make its frequency 5/2 times that of low C. Still too high. Bring it down another octave. That would make its frequency 5/4 times that of low C. That gives an E in the right octave for our scale (since 5/4 is between 1 and 2). With this approach, the major third from C to E has a frequency-multiplier of 5/4, which differs from that in Pythagoras' scale (which was 81/64), though not by much.
The interval from E to G is called a minor third. For the sake of the next section, we shall deduce its frequency-multiplier. We know frequency-multipliers for C to G (3/2) and for C to E (5/4, say). It follows that the frequency-multiplier for a minor third like E to G is 6/5, since 5/4 times 6/5 is 3/2.

Notes that were nearly but not quite the same

Returning to the topic of G♯ and A♭, we show why they were slightly different notes under one particular old tuning scheme. Suppose that there is a piece of music in C major that goes for a while into E major (when G♯ occurs) and also goes for a while into F minor (when A♭ occurs).
First we deduce a frequency-multiplier for C to G♯. Now C to E is a major third, and E to G♯ is another major third. Each major third has a frequency-multiplier of 5/4. Now 5/4 times 5/4 is 25/16, which must be the frequency-multiplier for C to G♯.
Next we deduce a frequency-multiplier for C to A♭. Now C to F is a fourth, and F to A♭ is a minor third. The frequency-multiplier for a fourth (4/3) followed by a minor third (6/5) is 4/3 times 6/5, which 8/5, the frequency-multiplier for C to A♭.
Since 25/16 is not quite equal to 8/5, it follows that in this system the notes G♯ and A♭ are not quite the same.

What was wrong with the old tuning(s)?

Nothing was *wrong* with them really.

Indeed, many musicians regarded a major chord like the C major chord (C, E and G) in a tuning with the frequency of G 3/2 times that of C and the frequency of E 5/4 times that of C as the most harmonious three-note chord possible, made up as it is only of notes found among the overtones of C (give or take an octave or two). This high regard for the major chord was the initial reason why many pieces in a minor key ended with a major chord.

Consequently, many musicians resisted the new Equal Temperament tuning system, calling it incorrect, and finding that even a two-note chord of C and E, say, was to them discordant in the new system. Actually, every chord involving other than octaves is dissonant to an extent. In the 19^{th} century, the science behind this was developed, and it became possible to quantify the dissonance of every interval. Those complaining about Equal Temperament had support from science, if we assume that dissonance is undesirable.

The old tuning systems, however, did tend to have features that may seem odd to us. For example, in any system that uses the "times 5/4" size of major third, and that uses ratios of whole numbers for frequency-multipliers for every interval, it is impossible to make the interval from C to D exactly the same as the interval from D to E, even though each of these intervals is called a tone.

Why change?

There were practical difficulties with the old "correct" tuning systems.

There was the complexity. If G♯ and A ♭ are different notes, then C to G♯ and C to A ♭ are different intervals, and so different names were needed for these intervals, and so on – see any music-theory textbook from around 1870.

There were limits to how much key-changing could be readily coped with. Consider a piece of music in which the key keeps changing to one whose keynote is one fifth above the previous keynote, where, as above, going up a fifth uses a frequency-multiplier of 3/2. This is an extreme example, rather than a practical one, perhaps, but its purpose is just to show the existence of a problem in a clear way. In this example, each new key really is a new key, one that has not occurred in the piece before. Ask a mathematician why. So there is no limit to the number of new keys, and therefore at some point we must run out of names for the keynotes, since we have at the very most only 21 note-names available (C, D, E, F, G, A and B, plus each of these with either a sharp or a flat) — unless we accept having *keys* like G double-sharp major. There would be problems too with how to write down the music to be played.
Pythagoras, by the way, knew the essence of this problem. He knew that if you picked a starting note and went up from it in octaves to create a sequence of notes, and if you went up from the same starting note in fifths (of the "times 3/2" kind) to create a second sequence of notes, then the only note shared by the two sequences of notes would be that starting note.

There were problems for the design of musical instruments. The human voice and instruments like the cello, violin or trombone can readily cope with G♯ and A ♭ being different notes, but the same is hardly true for the piano or pipe organ, say.

The idea behind equal temperament

We lose the connection between notes of a major scale and the overtones of the keynote.
We lose the mathematical simplicity of "times 3/2" for the change in frequency in going up a fifth.
We lose the mathematical simplicity of every frequency-multiplier being a ratio of whole numbers.
Instead, we have a different type of mathematical simplicity: we divide the interval called an

octave into twelve **equal** intervals called *semitones*. A tone will now be two semitones. A fifth will be seven semitones. A fourth will be five semitones. Every interval will be a whole number of semitones. Between consecutive notes in the major scale, the interval will be either one or two semitones.

The notes G♯ and A♭ are no longer different, but each is a semitone above G and a semitone below A. The problems mentioned in the previous section all disappear.

Equal temperament – the mathematics

Every interval has its own frequency-multiplier. So what is the frequency-multiplier for the kind of semitone that is exactly one twelfth of an octave?

Let **s** be the frequency-multiplier for a semitone.
Then the frequency-multiplier for a tone is $\mathbf{s} \times \mathbf{s} = \mathbf{s}^2$.
Similarly the frequency-multiplier for an interval of three semitones is $\mathbf{s} \times \mathbf{s} \times \mathbf{s} = \mathbf{s}^3$.
Similarly the frequency-multiplier for an interval of a fifth (7 semitones) is $\mathbf{s}^7$.
And the frequency-multiplier for an interval of an octave (twelve semitones) is $\mathbf{s}^{12}$.
But we already know that the frequency-multiplier for an interval of an octave is 2.
That gives us a little equation:

$$\mathbf{s}^{12} = 2$$

It follows that **s** is the number known as *the twelfth root of two*.

John MacNeill

Pythagoras' hemitone's frequency-multiplier is 256/243.